A TRIGONOMETRIC PROOF OF THE STEINER-LEHMUS THEOREM IN HYPERBOLIC GEOMETRY

KEIJI KIYOTA

Abstract. We give a trigonometric proof of the Steiner-Lehmus Theorem in hyperbolic geometry. Precisely we show that if two internal bisectors of a triangle on the hyperbolic plane are equal, then the triangle is isosceles.

1. Introduction

In 1844 [1], Steiner gave the first proof of the following theorem. If two internal bisectors of a triangle on the Euclidean plane are equal, then the triangle is isosceles. This had been originally asked by Lehmus in 1840, and now is called the Steiner-Lehmus Theorem. Since then, wide variety of proofs have been given by many people over 170 years. At present, at least 80 different proofs exist. See [6]. For example, in 2008, Hajja gave a short trigonometric proof in [2]. On the other hand, several proofs of this theorem in hyperbolic geometry ware given in [3], [4] and [5]. In this paper, we give a simple trigonometric proof in hyperbolic geometry based on the way of Hajja.

2. Steiner-Lehmus Theorem

Theorem. *If two internal bisectors of a triangle on the Hyperbolic plane are equal, then the triangle is isosceles.*

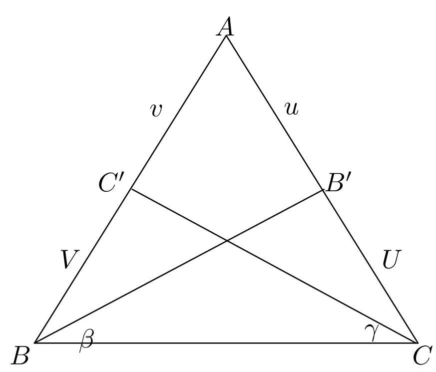

Figure 1.

Date: August 14, 2015.
2010 *Mathematics Subject Classification.* 51M09.
Key words and phrases. Steiner-Lehmus Theorem, hyperbolic geometry.

Proof. We consider a triangle ABC on the hyperbolic plane. See Figure 1. Let B' be intersection of the side AC and the internal bisectors of the angle B. Let C' be the intersection of the side AB and the internal bisector of the angle C. Then BB' and CC' are the internal bisectors of the angles B and C. Let a, b and c be the lenghs of the opposite sides of the angles A, B and C respectively. We set $\beta = B/2$, $\gamma = C/2$, $u = AB'$, $U = B'C$, $v = AC'$, and $V = C'B$.

We apply the sines theorem in hyperbolic geometry to the triangles ABC, BCC', ACC', CBB' and ABB' respectively, then we have the following.

$$\frac{\sinh a}{\sin A} = \frac{\sinh b}{\sin 2\beta} = \frac{\sinh c}{\sin 2\gamma} \tag{1}$$

$$\frac{\sinh CC'}{\sin 2\beta} = \frac{\sinh V}{\sin \gamma} \tag{2}$$

$$\frac{\sinh CC'}{\sin A} = \frac{\sinh v}{\sin \gamma} \tag{3}$$

$$\frac{\sinh BB'}{\sin 2\gamma} = \frac{\sinh U}{\sin \beta} \tag{4}$$

$$\frac{\sinh BB'}{\sin A} = \frac{\sinh u}{\sin \beta} \tag{5}$$

We assume $BB' = CC'$ and $C > B$, and lead to contradiction. Since the sum of the interior angles in a hyperbolic triangle is less than π, we have $B < C < \frac{\pi}{2}$, and so, $\sin B < \sin C$. In the following, we evaluate the magnitude relationship of u, v and U, V respectively.
By (2) and (4),

$$\frac{\sinh V}{\sin \gamma} \sin 2\beta = \frac{\sinh U}{\sin \beta} \sin 2\gamma$$
$$\frac{\sinh U}{\sinh V} = \frac{\sin \beta}{\sin \gamma} \frac{\sin 2\beta}{\sin 2\gamma}$$

By (1), we get $\frac{\sin 2\beta}{\sin 2\gamma} = \frac{\sinh b}{\sinh c}$, so we have the following.

$$\frac{\sinh U}{\sinh V} = \frac{\sin \beta}{\sin \gamma} \frac{\sinh b}{\sinh c}$$

Becaue of $\frac{\sin \beta}{\sin \gamma} < 1$ and $\frac{\sinh b}{\sinh c} < 1$, we get $\frac{\sin \beta}{\sin \gamma} \frac{\sinh b}{\sinh c} < 1$.Then we have $\sinh U < \sinh V$. Since the hyperbolic sine function is monotonically increasing, we conclude $U < V$.
Similarly, by (3) and (5),

$$\frac{\sinh v}{\sin \gamma} = \frac{\sinh u}{\sin \beta}$$
$$\frac{\sinh u}{\sinh v} = \frac{\sin \beta}{\sin \gamma} < 1$$

Therefore we get $\sinh u < \sinh v$, that is, $u < v$.

Now let us consider the ratio and difference of $\frac{\sinh b}{\sinh u}$ and $\frac{\sinh c}{\sinh v}$. First we consider the ratio.

$$\frac{\sinh b}{\sinh u} \Big/ \frac{\sinh c}{\sinh v} = \frac{\sinh b}{\sinh u} \frac{\sinh v}{\sinh c} = \frac{\sinh b}{\sinh c} \frac{\sinh v}{\sinh u}$$

We have the following by (1).

$$\frac{\sinh b}{\sinh c}\frac{\sinh v}{\sinh u} = \frac{\sin 2\beta}{\sin 2\gamma}\frac{\sin \gamma}{\sin \beta}$$

Here, we apply the double-angle formula to $\sin 2\beta, \sin 2\gamma$ respectively.

$$\frac{\sin 2\beta}{\sin 2\gamma}\frac{\sin \gamma}{\sin \beta} = \frac{2\sin\beta\cos\beta}{2\sin\gamma\cos\gamma}\frac{\sin\gamma}{\sin\beta} = \frac{\cos\beta}{\cos\gamma}$$

By assumption $\beta < \gamma$, we have $\cos\beta > \cos\gamma$. So $\frac{\cos\beta}{\cos\gamma} > 1$. Therefore we get the following result.

$$\frac{\sinh b}{\sinh u} > \frac{\sinh c}{\sinh v} \tag{6}$$

Next we consider the difference.

$$\frac{\sinh b}{\sinh u} - \frac{\sinh c}{\sinh v} = \frac{\sinh (U+u)}{\sinh u} - \frac{\sinh (V+v)}{\sinh v}$$

We apply the sum formula to $\sinh (U+u)$ and $\sinh (V+v)$ respectively.

$$\begin{aligned}\frac{\sinh (U+u)}{\sinh u} - \frac{\sinh (V+v)}{\sinh v} &= \frac{\sinh U\cosh u + \cosh U \sinh u}{\sinh u} - \frac{\sinh V \cosh v + \cosh V \sinh v}{\sinh v} \\ &= \frac{\sinh U}{\sinh u}\cosh u + \cosh U - \frac{\sinh V}{\sinh v}\cosh v + \cosh V\end{aligned}$$

By (4), (5) and (2), (3), $\frac{\sinh U}{\sinh u} = \frac{\sin A}{\sin 2\gamma}$ and $\frac{\sinh V}{\sinh v} = \frac{\sin A}{\sin 2\beta}$ hold, and so, we have the following.

$$\frac{\sinh U}{\sinh u}\cosh u+\cosh U-\frac{\sinh V}{\sinh v}\cosh v+\cosh V = \frac{\sin A}{\sin 2\gamma}\cosh u+\cosh U-\frac{\sin A}{\sin 2\beta}\cosh v+\cosh V$$

Moreover we get the following by (1).

$$\frac{\sin A}{\sin 2\gamma}\cosh u+\cosh U-\frac{\sin A}{\sin 2\beta}\cosh v+\cosh V = \frac{\sinh a}{\sinh c}\cosh u+\cosh U-\frac{\sinh a}{\sinh b}\cosh v-\cosh V$$

By $\sinh c > \sinh b$, we have $\frac{\sinh a}{\sinh b} > \frac{\sinh a}{\sinh c}$. And $\cosh v > \cosh u$ and $\cosh V > \cosh U$ by $u < v$ and $U < V$. Therefore we get the following.

$$\frac{\sinh a}{\sinh c}\cosh u + \cosh U - \frac{\sinh a}{\sinh b}\cosh v - \cosh V < 0$$

Eventually we conclude the following result.

$$\frac{\sinh b}{\sinh u} < \frac{\sinh c}{\sinh v} \tag{7}$$

A contradiction is led by (6) and (7). □

References

[1] J. Steiner, Elementare Lösung einer Aufgabe über das ebene und sphärische Dreieck, J. Reine Angew. Math. **28** (1844), 375–379.

[2] M. Hajja, A short trigonometric proof of the Steiner-Lehmus theorem, Forun Geom. **8** (2008), 39–42.

[3] N. Sönmez, Trigonometric proof of Steiner-Lehmus theorem in hyperbolic geometry, KoG **12** (2008), 35–36.

[4] C. Barbu, Trigonometric proof of Steiner-Lehmus theorem in hyperbolic geometry, Acta Univ. Apulensis Math. Inform. No. 23 (2010), 63–67.

[5] O. Demirel, The theorems of Urquhart and Steiner-Lehmus in the Poincaré ball model of hyperbolic geometry, *Mat. Vesnik.* **63** (2011), 263–274.

[6] V. Pambuccian, H. Struve, and R. Struve, The Steiner-Lehmuus theorem and "triangles with congruent medians are isosceles" hold in weak geometries, preprint, arXiv:1501.01857.

Graduate School of Integrated Basic Sciences, Nihon University, 3-25-40 Sakurajo-sui, Setagaya-ku, Tokyo 156-8550, Japan

E-mail address: `ta15039@educ.chs.nihon-u.ac.jp`